Coherentist vs. foundationalist belief revision: bringing them together

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Abstract

In the belief change literature, belief revision operations apply to belief bases. When these bases are closed under logical consequence they are called belief sets. The main difference is that the latter satisfy the principle of extensionality: when two logically equivalent belief bases are revised by the same new piece of information then the outcomes are again logically equivalent. The main argument that was brought forward against extensionality is that while the two belief bases $\{p, q\}$ and $\{p, p \to q\}$ are 'statically equivalent', they are not 'dynamically equivalent', in the sense that they have to be revised differently. We argue that this non-equivalence should be made formal by investigating revision for logics where these two belief bases fail to be equivalent. We argue that logics of strict implication are suitable candidates for such logics and propose a particular distance-based revision operation that lifts Dalal's operation from a distance between valuations to a distance between pointed S5 models.

1 Introduction

The work of Alchourrón, Gärdenfors and Makinson (AGM) on belief revision operations is in terms of belief sets: sets of formulas that are closed under logical theorems and modus ponens. Such sets are infinite as soon as the set of propositional variables is so. However, it has been argued in the philosophical as well as in the AI literature that this is not a realistic hypothesis, both in the case of human agents and in the case of computers, whose memory can only contain finite sets of formulas that fail to be closed under theorems and modus ponens. Such sets are called belief bases. One of the most appealing arguments for belief bases is that while the belief bases $\{p, q\}$ and $\{p, p \to q\}$ are logically equivalent in classical propositional logic, they should however not be revised in the same way: the revision of the former by $\neg q$ should result in $\{p, \neg q\}$ because p and q are independent.[1] In contrast, so the argument goes, the revision of $\{p, p \to q\}$ by $\neg q$ should lead to $\{p \to q, \neg q\}$. The latter is logically incomparable with $\{p, \neg q\}$, in the sense that neither implies the other.

A variant of the argument can be formulated in terms of material equivalence [Han17]:

> As an example, the two belief bases $\{p, q\}$ and $\{p, p \leftrightarrow q\}$ have the same logical closure. They are therefore *statically equivalent*, in the sense of representing the same beliefs. On the other hand, the following example shows that they are not *dynamically equivalent* in the sense of behaving in the same way under operations of change. They can be taken to represent different ways of holding the same beliefs. Let p denote that the Liberal Party will support the proposal to subsidize the steel industry, and let q denote that Ms. Smith, who is a liberal MP, will vote in favour of that proposal. Abe has the basic beliefs p and q, whereas Bob has the basic beliefs p and $p \leftrightarrow q$. Thus, their beliefs (on the belief set level) with respect to p and q are the same. Both Abe and Bob receive and accept the information that p is wrong, and they both revise their belief states to include the new belief that $\neg p$. After that, Abe has the basic beliefs $\neg p$ and q, whereas Bob has the basic beliefs $\neg p$ and $p \leftrightarrow q$. Now, their belief sets are no longer the same. Abe believes that q whereas Bob believes that $\neg q$.

The property of 'dynamical non-equivalence' of $\{p, q\}$ and $\{p, p \to q\}$ as well as that of $\{p, q\}$ and $\{p, p \leftrightarrow q\}$ is only exploited by belief base revision theories, but it is not explained: a natural question to ask is whether one can give a logical foundation to the distinction. Intuitively, the material implication $p \to q$ is too weak to adequately capture the 'if-then' link between p and q. Actually Bob's belief that $p \leftrightarrow q$ (that Ms. Smith will conform to the party decision) has

[1] This can be motivated by Parikh's idea of language splitting [KM07, Par99].

a dispositional nature: Bob's belief that that $p \leftrightarrow q$ is not only about here and now, that is, when it is the case that the party supports the proposal, but also about the situation where the party does not support the proposal.

We conclude from this discussion that material implication and material equivalence are too weak in order to account for 'dynamical non-equivalence' that people are after: they should be replaced by stronger connectives. This leads us to examine logics with an implication connective $\rightarrow$ such that $\{p, q\}$ and $\{p, p \rightarrow q\}$ fail to be equivalent: can such logics be equipped with revision operations that satisfy extensionality?

It is clear that the most well-known substructural logics are of no help: to witness, our two belief bases are logically equivalent both in intuitionistic logic and in linear logic. Relevance logics are more interesting: the formulas $p \wedge q$ and $p \wedge (p \rightarrow q)$ typically fail to be equivalent there. These logics however have a paraconsistent nature: basically every set of formulas is consistent, and it is not clear how to design a revision operation that avoids collapsing into expansion. As far as we know this issue was only addressed in the two papers [RS95, LL96]. It seems fair to say that this research avenue was not too fruitful; in any case, it was not pursued beyond these two papers, possibly because the underlying relevance logics are fairly complex and because the resulting revision operations depart significantly from the AGM framework.

We here take another, more promising research avenue that, as far as we know, has not been tried yet: the logic of strict implication, noted "$>$" [LL59, HC68].[2] We design a belief revision operation that is based on a generalisation of the Hamming distance to S5 Kripke models. When neither the belief base nor the input contains strict implications then our account amounts to Dalal's belief revision operation [Dal88]. Our distance-based operation revises $\{p, q\} *^{\mathsf{strict}} \neg q$ and $\{p, p \rightarrow q\} *^{\mathsf{strict}} \neg q$ differently and in the expected way.

2 The logic of strict implication

Strictly speaking, there is not just one logic of strict implication, but a whole family of such logics, depending on the class of frames under consideration. We here choose S5 frames, which is mainly motivated by their simplicity.

The language of strict implication is defined by the following grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi > \varphi$$

where p ranges over the set of propositional variables PVar. Material implication $\varphi \rightarrow \psi$ is defined to be an abbreviation of $\neg\varphi \vee \psi$. A *classical propositional formula* is a formula in which no strict implication occurs.

A *pointed S5-model* is a couple $M = (W, w)$ where W is a non-empty subset of the set of possible valuations 2^{PVar} and $w \in W$. In examples we drop the set parentheses of a possible world and write e.g. p instead of $\{p\}$ and pq instead of $\{p, q\}$. So the set of possible worlds $\{\{p\}, \{p, q\}\}$ is written $\{p, pq\}$. Moreover, we identify the actual world by underlining it. For example, $\{\underline{\emptyset}, pq\}$ stands for the pointed S5-model $\langle\{\emptyset, pq\}, \emptyset\rangle$.

The truth condition for strict implication is:

$$W, w \Vdash \varphi > \psi \text{ if for every } w' \in W : W, w' \nVdash \varphi \text{ or } W, w' \Vdash \psi .$$

The other truth conditions are the usual ones. The set of pointed models of φ is

$$||\varphi|| = \{M : M \Vdash \varphi\} .$$

For example, the set of models of $p \wedge (p > q)$ is

$$||p \wedge (p > q)|| = \{(W, w) : w \in W, \ p \in w \text{ and for every } w' \in W : p \notin w' \text{ or } q \in w\}.$$

When PVar is $\{p, q\}$ then $||p \wedge (p > q)||$ equals $\{\{\underline{pq}, q, \emptyset\}, \{\underline{pq}, q\}, \{\underline{pq}, \emptyset\}, \{\underline{pq}\}\}$. More generally we have:

Proposition 1 *If φ is a classical propositional formula then*

$$||\varphi|| = \{(W, w) : W \subseteq 2^{\mathsf{PVar}} \text{ and } w \in W \cap ||\varphi||_{\mathsf{PC}}\},$$

where $||\varphi||_{\mathsf{PC}} \subseteq 2^{\mathsf{PVar}}$ is the set of classical propositional models of φ.

Note that both $(p \wedge (p > q)) \rightarrow (p \wedge q)$ and its strict variant $(p \wedge (p > q)) > (p \wedge q)$ are both valid. In contrast, neither $(p \wedge q) \rightarrow (p \wedge (p > q))$ nor $(p \wedge q) > (p \wedge (p > q))$ are valid. It is exactly this non-equivalence of $p \wedge (p > q)$ and $p \wedge q$ that makes the logic of strict implication an interesting basic logic for belief revision.

[2] More generally, revision based on modal logics was not particularly studied in the literature, with the exception of Hansson and Fuhrmann's finding that many of the postulates make little sense when the language has modal operators that have an autoepistemic nature: then e.g. preservation cannot be maintained [Han99, Fuh89].

3 Generalising the Hamming distance

We are going to view the set of accessible worlds W of a pointed model (W,w) as a first circle of possible 'fallback worlds' in a system of spheres: when a new piece of information is false at the actual world w then the 'first try' is to replace w by one of the other worlds in W. It is only when none of the elements of W satisfies the new piece of information that we have to modify W. In both cases we are going to determine the resulting model by means of a distance measure.

Revisions operations can be built on arbitrary orderings or distances. We here start from a particular distance between valuations: the Hamming distance. That distance is defined between valuations, and we are going to lift it to pointed S5-models.

The Hamming distance between two sets of propositional variables $w, u \in 2^{\mathsf{PVar}}$ is the cardinality of their symmetric difference:

$$d(w,u) = |(w \setminus u) \cup (u \setminus w)|.$$

We lift it to a distance between a valuation w and a set of valuations $U \subseteq 2^{\mathsf{PVar}}$ by defining

$$D(w,U) = \min\{d(w,u) \ : \ u \in U\}.$$

Observe that $D(w,U)$ equals 0 exactly when $w \in U$.

Let $\leq_{\mathsf{lex}}$ be the *lexicographic ordering* on couples of integers, defined as: $\langle k,n\rangle \leq_{\mathsf{lex}} \langle k',n'\rangle$ iff either $k \leq k'$, or $k = k'$ and $n \leq n'$. This ordering allows us to define the distance between two pointed models (W,w) and (U,u) as a couple of integers.

Definition 1 *For two pointed S5 models (W,w) and (U,u):*

$$\Delta((W,w),(U,u)) = \Big\langle \sum_{w' \in W} D(w',U) + \sum_{u' \in U} D(u',W), d(w,u) \Big\rangle.$$

We finally lift the above $\Delta((W,w),(U,u))$ to a distance between a set of pointed models $\mathcal{M}$ and a pointed model N:

$$\Delta(\mathcal{M},N) = \min_{\leq_{\mathsf{lex}}}\{\Delta(M,N) \ : \ M \in \mathcal{M}\}.$$

Fact 1 *Let $M_1 = (W_1,w_1)$ and $M_2 = (W_2,w_2)$ be two pointed S5 models. Then*

- *$w_1 = w_2$ if and only if $\Delta(M_1,M_2) = \langle k,0\rangle$ for some k;*
- *$W_1 = W_2$ if and only if $\Delta(M_1,M_2) = \langle 0,n\rangle$ for some n.*

Example 1 *We have:*

$$\Delta(\|\neg q\|, \|p \wedge q\|) = \langle 0,1\rangle \tag{1}$$
$$\Delta(\|\neg p\|, \|p \wedge (p > q)\|) = \langle 0,1\rangle \tag{2}$$
$$\Delta(\|\neg q\|, \|p \wedge (p > q)\|) = \langle 0,2\rangle \tag{3}$$

For item (1), observe that, first, the distance cannot be $\langle 0,0\rangle$ because the intersection of $\|\neg q\|$ and $\|p \wedge (p > q)\|$ is empty; second, for the pointed models $\{\underline{p}, pq\} \in \|\neg q\|$ and $\{p, \underline{pq}\} \in \|p \wedge q\|$ we have $\Delta(\{\underline{p}, pq\}, \{p, \underline{pq}\}) = \langle 0,1\rangle$.

For item (2), observe that, again, the intersection of $\|\neg p\|$ and $\|p \wedge (p > q)\|$ is empty and that for the pointed models $\{\underline{q}, pq\} \in \|\neg p\|$ and $\{q, \underline{pq}\} \in \|p \wedge (p > q)\|$ we have $\Delta(\{\underline{q}, pq\}, \{q, \underline{pq}\}) = \langle 0,1\rangle$.

For item (3), observe that:

1. *the intersection of $\|\neg q\|$ and $\|p \wedge (p > q)\|$ is empty;*
2. *for all $(U,u) \in \|\neg q\|$ such that $\Delta(\{(U,u)\}, \|p \wedge (p > q)\|) = 0$ we have $(U,u) \Vdash \neg p \wedge \neg q$; and for all $(W,w) \in \|p \wedge (p > q)\|$ we have $(W,w) \Vdash p \wedge q$; it follows that $d(u,w) \geq 2$.*
3. *We finally have for the pointed models $\{\underline{\emptyset}, pq\} \in \|\neg q\|$ and $\{\emptyset, \underline{pq}\} \in \|p \wedge (p > q)\|$ that $d(\emptyset, pq) = 2$.*

Putting these three observations together we obtain $\Delta(\|\neg q\|, \|p \wedge (p > q)\|) = \langle 0,2\rangle$.

4 A distance-based revision operation

Distance-based revision operations were studied by Lehman et al. and Ben-Naim [LMS01, Ben06]. We take over their semantic definition and base it on the lexicographic ordering of distances between pointed models that we have introduced.

To warm up we recall the definition of Dalal's belief revision operation. Let φ and μ be formulas of classical propositional logic, i.e., without strict implications. Then:

$$\varphi *^{\mathsf{dalal}} \mu = \{u \in ||\mu||_{\mathsf{PC}} \ : \ \text{there is no } u' \in ||\mu||_{\mathsf{PC}} \text{ such that } d(||\varphi||_{\mathsf{PC}}, u') < d(||\varphi||_{\mathsf{PC}}, u).$$

The next definition generalises Dalal's operation to arbitrary formulas φ and μ of the language of strict implication.

Definition 2 *For two formulas φ and μ of the language of strict impliation:*

$$\varphi *^{\mathsf{strict}} \mu = \{N \in ||\mu|| \ : \ \textit{there is no } N' \in ||\mu|| \textit{ such that } \Delta(||\varphi||, N') <_{\mathsf{lex}} \Delta(||\varphi||, N)\}.$$

Example 2 *We have:*

$$p \land q *^{\mathsf{strict}} \neg q = ||p \land \neg q|| \tag{4}$$
$$p \land (p \succ q) *^{\mathsf{strict}} \neg q = ||\neg q \land (p \succ q)|| \tag{5}$$
$$p \land (p \succ q) *^{\mathsf{strict}} \neg p = ||\neg p \land q \land (p \succ q)|| \tag{6}$$
$$p \succ q *^{\mathsf{strict}} q \succ p = ||(p \succ q) \land (q \succ p)|| \tag{7}$$
$$p \land \neg q *^{\mathsf{strict}} p \succ q = ||(p \succ q) \land (\neg q \lor p)|| \tag{8}$$

Items 4 and 5 illustrate that $*$ satisfies the intuitions we have put forward in the introduction. Item 6 is worth discussing: one might expect that q is preserved because the p 'causing' q via $p \succ q$ has gone. This however fails to hold: the strict implication $p \succ q$ does not capture more demanding, causality-alike concepts. Example 7 is in accordance with the preservation postulate: as $p \succ q$ and $q \succ p$ are consistent with each other, the revision of the former by the latter is equivalent to their conjunction. For Example 8, the minimal lexicographic distance from $||p \land q||$ to the elements of $||p \succ q||$ is $(1,1)$, hence the closest actual world after the revision is either $\{p, q\}$ or $\emptyset$.

It follows from Lehmann and Schlechta's results that $*$ satisfies the AGM postulates, plus some more. Just as Dalal's revision operation, $*$ satisfies Parikh's language splitting postulate [Par99, KM07].

When strict implication neither occurs in the belief base nor in the input then our revision operation boils down to Dalal's belief revision operation $*^{\mathsf{dalal}}$.

Proposition 2 *If φ and μ are classical propositional formulas then*

$$\varphi *^{\mathsf{strict}} \mu = \{(U,u) \ : \ U \subseteq 2^{\mathsf{PVar}} \textit{ and } u \in U \cap (\varphi *^{\mathsf{dalal}} \mu).$$

The proof uses Proposition 1. For example, the revision of $p \land (p \to q)$ by $\neg q$ equals the revision of $p \land q$ by $\neg q$, namely the set of all pointed models where p is true and q is false in the actual world.

5 From pseudometrics to metrics

Whereas the Hamming distance is widely used in the literature for measuring distance between worlds, distances between models can be defined in other ways. Any modification of the distance measure Δ gives rise to different revision operators. Some alternatives can be found in [CKdLM16].

The distance D that we have introduced in section 3 is actually only a pseudodistance: it does not satisfy the triangle inequality axiom and cannot be classified as a metric for that reason. In the literature, it is proven that pseudodistances are enough to define revision operators that have good properties [LMS01]. It might however be of interest to explore the revision operators that arise when a true metric is employed. We now introduce a metric $D_{\mathtt{inj}}$ and study the behavior of the resulting operator. The general case of this definition was proposed in [SCI20].

For $W, U \subseteq 2^{\mathsf{PVar}}$, let $\mathtt{Dmax}(W, U)$ be $\max\{d(w, u) : w \in W \text{ and } u \in U\}$.

Definition 3 *For $W, U \subseteq 2^{\mathsf{PVar}}$ such that $|W| \le |U|$ and any injection $f : W \to U$:*

$$D_{inj}(X,Y) = \min\left\{\left(\sum_{x \in X} d(x, f(x))\right) + (||W| - |U|| * \mathtt{Dmax})\right\}.$$

Definition 4 *For (W, w) and (U, u) pointed S5 models:*

$$\Delta_{inj}(W,w)(U,u) = \Big\langle D_{inj}(W, U),\ d(w, u)\Big\rangle.$$

We denote by $*^{\mathsf{inj}}$ the revision operator resulting from the replacement of Δ by Δ_{inj} in Definition 2. We illustrate the difference with the revision operator of Section 4 by means of an example.

Example 3 *Let us take up item 7 of Example 2 where we revised $p > q$ by $\neg(p > q)$. We show that in this case, the output with $*^{\mathsf{inj}}$ differs. Observe first that the distance between models of $p > q$ and $\neg(p > q)$ cannot be zero. Then for the pointed models $\{\underline{q},\ pq\} \in \|p > q\|$ and $\{\underline{q},\ p\} \in \|\neg(p > q)\|$ we have:*

$$\Delta_{inj}\Big(\{\underline{q}, pq\},\ \{\underline{q}, p\}\Big) = \langle 1, 0\rangle\ ;$$
$$\Delta\Big(\{\underline{q}, pq\}, \{\underline{q}, p\}\Big) = \langle 2, 0\rangle\ .$$

Now consider the following two pointed models $\{\underline{pq}\} \in \|p > q\|$ and $\{\underline{p},\ pq\} \in \|\neg(p > q)\|$. We have:

$$\Delta_{inj}\Big(\{\underline{pq}\}, \{\underline{pq},\ p\}\Big) = \langle 2, 0\rangle\ ;$$
$$\Delta\Big(\{\underline{pq}\}, \{\underline{pq},\ p\}\Big) = \langle 1, 0\rangle\ .$$

Therefore $\{\underline{q},\ p\}$ belongs to $p > q *^{\mathsf{inj}} \neg(p > q)$ while it does not belong to $p > q *^{\mathsf{strict}} \neg(p > q)$. Similarly, $\{\underline{pq},\ p\}$ does not belong to $p > q *^{\mathsf{inj}} \neg(p > q)$ while it does belong to $p > q *^{\mathsf{strict}} \neg(p > q)$.

6 Related work

Caridroit et al. have proposed an extension of the Hamming distance to KD45-models [CKdLM16]. These contributions take a doxastic perspective where accessible worlds model an agent's doxasitc state. They consider that two pointed models (W,w) and (U,u) agreeing on the actual world (in our terms: $w = u$) have smaller distance than models disagreeing on the actual world. Our contribution takes the opposite view: agreement on the actual world is less important than agreement on the set of accessible worlds, as embodied in our lexicographic ordering on pointed models $\le_{\mathsf{lex}}$. As our distance differs from Caridroit et al.'s, the resulting revision operation is different from theirs.

Our revision operation revises the four sets $\{p, q\}$, $\{p, q, p \wedge q\}$, $\{p \wedge q\}$ and $\{q \wedge p\}$ in the same way. In contrast, many belief base revision operations allow for different outcomes. For example, in many approaches we have that the revision of $\{p, q\}$ by $\neg q$ is $\{p, \neg q\}$, while the revision of $\{p \wedge q\}$ by $\neg q$ is $\{\neg q\}$. However, it seems to us that such differences are much more difficult to justify than the difference between $\{p, q\} *^{\mathsf{strict}} \neg q$ and $\{p, p > q\} *^{\mathsf{strict}} \neg q$ that has motivated our approach.

In a spirit similar to ours, Konieczny et al. proposed to explain that the belief bases $\{p, q\}$ and $\{p \wedge q\}$ are revised differently [KLM05]. They did so by giving to the set comma the status of a logical connective.

7 Discussion and perspectives

Beyond the AGM postulates, our strict implication-based belief revision operation satisfies Lehmann et al.'s postulates for distance-based belief revision. It also satisfies Parikh's postulate for language splitting. It would be interesting to have a complete set of postulates characterising our operation. This is however certainly not easy to achieve: our operation extends Dalal's belief revision operation, for which as far as we know no characterisation exists in the literature.

Our work could be extended in two directions. First, we could consider logics of strict implication other than S5. This requires a careful design of the distance measure. More generally, strict implication can be viewed as a particular conditional operator, and in many conditional logics the formulas $p \wedge q$ and $p \wedge (p > q)$ fail to be equivalent.[3]

Second, we could move from the Hamming distance to the set-inclusion based ordering underlying Satoh's revision operation [Sat88] and Winslett's update operation [Win88, Win90].

In logics of strict implication, the alethic modality $\Box\varphi$ can be defined as $\top > \varphi$; similarly, $\Diamond\varphi$ can be defined as $\neg(\varphi > \bot)$. The other way round, one may as well take "$\Box$" as primitive and define $\varphi > \psi$ to be $\Box(\neg\varphi \vee \psi)$. One may then conveniently express integrity constraints by prefixing them by $\Box$. This relates to belief revision and merging operations under integrity constraints [KP02]; however, our operation allows for the revision of integrity constraints themselves, either through input that is inconsistent with the constraints, or through a new integrity constraint.

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[3] Precisely, the two formulas are logically equivalent in those of Lewis's sphere systems [Lew73] satisfying both weak and strong centering, that is, the actual world is the only element of the innermost sphere. Such models validate the axiom of conditional modus ponens $(p \wedge (p > q)) \rightarrow q$ and the axiom of conjunctive sufficiency $(p \wedge q) \rightarrow (p > q)$.

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